Stochastic Optimization Model for Long Term Debt Issuance

Office of Debt Management

November 14th, 2012
This presentation offers a high level description of the implementation of ODM’s (Office of Debt Management) stochastic optimization model for long term debt issuance.

Outline
- Terminology
- Model Overview
- The 5-step of the Model implementation:
  - Step 1: Projected paths
  - Step 2: Issuance strategies
  - Step 3: Debt mechanics process
  - Step 4: Function approximation
  - Step 5: Optimization
- Results
- Appendix: VAR model
Terminology

- **Financing needs** – Amount needed for a given fiscal year, (i) to finance the excess of outlays (excluding interest costs) over receipts, often referred to as “primary deficit”, and (ii) to service existing and maturing debt.

- **Projected paths** –
  - *Deterministic path*: point estimates of economic variables in the future. Default path is OMB (Office of Management and Budget) forecast, but we can use any user-generated forecast.
  - *Stochastic paths*: use a vector autoregressive model of economic variables with mean reversion to the deterministic path to simulate a set of economic paths.

- **Issuance Strategies (or “financing strategies”)** – A debt financing strategy that specifies the issuance proportion of Bills, Notes, Bonds and TIPS according to an issuance schedule for a projection horizon.

- **Debt Mechanics process** – A set of fixed income calculations that translate an issuance strategy over a projected path into a debt issuance size at each step.

- **Performance metrics** – Numeric measurement of the performance of an issuance strategy over a set of projected path of economic variables. The measurement includes total interest cost, terminal debt outstanding, weighted-average-maturity (WAM) etc.

- **Function approximation** – A function approximation technique to fit each of the pre-defined performance metrics into a functional form (to be used in optimization later) of the issuance strategy variables.

- **Optimization** – A technique to find an optimal debt issuance strategy for an objective function subject to a set of constraints.
The Model Overview

- ODM’s stochastic optimization model for long term debt issuance allows ODM to
  - Evaluate the performance of various debt issuance strategies and identify an optimal strategy based on different performance metrics.
  - Add additional performance metrics as constraints or optimization objectives in the future.
  - Undertake “What-If” analyses to explore the impact of various economic variables (e.g. interest rates, CPI levels, outlays and receipts) upon the optimal issuance strategies.

- This approach devises an optimal strategy under various economic scenarios, focusing on dealing with the dispersion of the potential economic outcome.

- We will discuss the results of the approach and their implications.
The Long Term Optimization model is implemented as a 5-step process:

1. Projected paths of economic variables:
   - Given a deterministic path, generate a set of stochastic paths.

2. Issuance strategy
   - Generate a set of issuance strategies with min/max and gradient constraints.

3. Debt Mechanics – Performance Metrics Generation
   - An algorithm of translating issuance strategies into specific debt issuance sizes (dollar amount) at each step of each projected path.
   - Measure each issuance strategy on projected paths with seven performance metrics.

4. Function Approximation
   - Devise a quadratic function of the issuance strategy variables to approximate the value of its associated performance metrics.

5. Optimization
   - Optimize an objective function over issuance strategy variables subject to constraints.
   - Both objective function and constraints are linear combinations of performance metrics.
Step 1: Projected Paths

- Generate a set of realistic paths of economic variables.
- The set of paths are consistent with the user-specific projection.
- The economic variables include interest rates dynamics, CPI levels, and financing needs.

- We have two kinds of projected paths:
  - Deterministic path: point estimates economic variables from OMB or any user input.
  - Stochastic paths: use a VAR (Vector Autoregressive) model of with mean reversion to the deterministic path to simulate a set of paths.

- VAR Model is built on three categories of variables
  - Interest rate factors (nominal rates and BEI (breakeven inflation: spread between nominal and real)
  - Receipts and Outlays
  - Macro factors (GDP and CPI levels)
- VAR Model captures co-movements of all these variables.

- The mission of this VAR model is not to produce an independent set of economic forecasts. Whereas forecasting focuses on the mean outcome, this simulation model focuses on the dispersion of potential outcomes.

- See Appendix A for the details of the model.
Step 2: Issuance Strategies

- An issuance strategy is modeled as the proportions of four Treasury securities: Bills, Notes, Bonds, and TIPS. The proportions sum up to be 1.
- Mathematically, an issuance strategy is represented as a vector of 9 variables (3 proportions at 3 time spots):
  - 3 proportions: proportions of bills, notes and bonds. (The TIPS proportion is implicitly determined since all proportions summing up to be 1.)
  - 3 time spots: year 2, 5, and 10. Proportions for any remaining years before year 10 are obtained via linear interpolation. Proportions for years beyond year 10 remain unchanged from year 10.
- We set the min/max and year over year gradients constraints on issuance strategies.
- Generate a permissible issuance strategy (passing all constraints) using uniformly distributed random numbers.
- Generate sufficient issuance strategies to cover the span of permissible strategies.
- An example of an issuance strategy with min/max and gradient constraints:

<table>
<thead>
<tr>
<th>Year</th>
<th>Proportion</th>
<th>Bills</th>
<th>Notes</th>
<th>Bonds</th>
<th>TIPS</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td>0.40</td>
<td>0.50</td>
<td>0.05</td>
<td>0.05</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>0.44</td>
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<td>0.03</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>0.48</td>
<td>0.41</td>
<td>0.10</td>
<td>0.01</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>0.47</td>
<td>0.44</td>
<td>0.08</td>
<td>0.01</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>0.47</td>
<td>0.48</td>
<td>0.05</td>
<td>0.00</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>0.46</td>
<td>0.51</td>
<td>0.03</td>
<td>0.00</td>
</tr>
<tr>
<td>6</td>
<td></td>
<td>0.44</td>
<td>0.51</td>
<td>0.04</td>
<td>0.02</td>
</tr>
<tr>
<td>7</td>
<td></td>
<td>0.42</td>
<td>0.51</td>
<td>0.06</td>
<td>0.04</td>
</tr>
<tr>
<td>8</td>
<td></td>
<td>0.41</td>
<td>0.52</td>
<td>0.07</td>
<td>0.05</td>
</tr>
<tr>
<td>9</td>
<td></td>
<td>0.39</td>
<td>0.52</td>
<td>0.09</td>
<td>0.07</td>
</tr>
<tr>
<td>10</td>
<td></td>
<td>0.37</td>
<td>0.52</td>
<td>0.10</td>
<td>0.09</td>
</tr>
</tbody>
</table>
Step 3: Debt Mechanics – Metrics Generation

- Debt Mechanics process is a set of rules, configurations and calculations that translate an issuance strategy (proportion levels) into a debt issuance size (dollar amount) of each security at each time step.
  - Security issuance size (dollar amount) remain unchanged within the same fiscal year. For example, if we start with $40 BN for a 4-week bill at the beginning of year 0, then we can the size for the remaining of year 0.
  - The proportions of each security within their security tranche remain unchanged throughout the whole projection horizon. For example, if we start with the proportions of bills: 4-week, 13-week, 26-week, and 52-week as 25% each of the total bill issuance, then this ratio will remain unchanged throughout the projection horizon.

- The calculations involved are typical fixed income portfolio calculations like coupon payments, maturing debt, interest cost, new issuance, financing needs and cash balances.

- Measure each issuance strategy using the following seven performance metrics over projected paths (default: 500 paths).
  - Total interest cost over the next 10 years
  - Total interest cost over the next 20 years.
  - Lumpiness of maturity (quarterly maturity profile)
  - Terminal WAM (Weighted Average Maturity)
  - Average WAM (Weighted Average Maturity)
  - Interest cost volatility
  - Terminal debt outstanding

- For each issuance strategy of each performance metric over the projected paths, we calculate three statistics: mean, standard deviation, and CVaR (Conditional VaR). See next page for details for performance metrics and their associated statistics.
We calculate three statistics of each metric for every strategy over the projected paths:

- Mean: the mean of each metric over all projected paths.
- Standard deviation: the standard deviation of each metric over all projected paths.
- CVaR: the conditional VaR (at 90% confidence level) of each metric; or the average of the worst 10% of the metrics; this measures the tail risk of the associated metric.
- In total, there are 21 = (7 metrics * 3 statistics) numeric performance metrics associated with each issuance strategy.
Step 4: Function Approximation

- Each issuance strategy has 21 numeric performance metrics (3 statistics * 7 performance metrics).
- For example, for issuance strategy #1 to 5, each has three statistics of 10-year interest cost over all paths.

<table>
<thead>
<tr>
<th>Issuance Strategy #</th>
<th>w2,1</th>
<th>w2,2</th>
<th>w2,3</th>
<th>w5,1</th>
<th>w5,2</th>
<th>w5,3</th>
<th>w10,1</th>
<th>w10,2</th>
<th>w10,3</th>
<th>Int Cost 10yrs Mean ($BN)</th>
<th>Int Cost 10yrs Std. Dev. ($BN)</th>
<th>Int Cost 10yrs CVaR ($BN)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.403</td>
<td>0.495</td>
<td>0.076</td>
<td>0.344</td>
<td>0.580</td>
<td>0.013</td>
<td>0.221</td>
<td>0.675</td>
<td>0.023</td>
<td>5,047.5</td>
<td>117.1</td>
<td>5,255.9</td>
</tr>
<tr>
<td>2</td>
<td>0.546</td>
<td>0.327</td>
<td>0.029</td>
<td>0.244</td>
<td>0.674</td>
<td>0.013</td>
<td>0.342</td>
<td>0.559</td>
<td>0.017</td>
<td>4,928.2</td>
<td>116.3</td>
<td>5,134.9</td>
</tr>
<tr>
<td>3</td>
<td>0.374</td>
<td>0.526</td>
<td>0.018</td>
<td>0.525</td>
<td>0.419</td>
<td>0.019</td>
<td>0.490</td>
<td>0.458</td>
<td>0.036</td>
<td>4,971.7</td>
<td>114.6</td>
<td>5,176.1</td>
</tr>
<tr>
<td>4</td>
<td>0.373</td>
<td>0.532</td>
<td>0.052</td>
<td>0.524</td>
<td>0.352</td>
<td>0.079</td>
<td>0.587</td>
<td>0.369</td>
<td>0.038</td>
<td>5,117.0</td>
<td>116.4</td>
<td>5,324.1</td>
</tr>
<tr>
<td>5</td>
<td>0.218</td>
<td>0.665</td>
<td>0.067</td>
<td>0.477</td>
<td>0.354</td>
<td>0.097</td>
<td>0.429</td>
<td>0.525</td>
<td>0.037</td>
<td>5,047.9</td>
<td>115.6</td>
<td>5,253.4</td>
</tr>
</tbody>
</table>

- $w(i,j)$ is the issuance proportion in $i$-th year and $j$-th tranche, e.g., $w(2,1)$ is the proportion in the 2nd year for 1st tranche. The tranches are:
  - 1st tranche: Bills
  - 2nd tranche: Notes
  - 3rd tranche: Bonds
  - 4th tranche: TIPS (which is calculated as the difference between 1 and the sum of first three tranches.)
Step 4: Function Approximation Cont’d

- We use a quadratic function for function approximation: each metric is fit into a function of 9 variables: $w(i,j)$.
- To continue the example given above: the average 10-year interest cost is approximated as a quadratic function $f$ of the 9 variables $w(i,j)$.

$$ f = $$

- Term $(1, 1)$ here is the constant term, with a value of 2,510.3 in this example.
- The last column contains coefficients of linear terms $w(i,j)$; e.g. the coefficient of $w_2,1$ is 1,104.7.
- Diagonal terms are coefficients on $w(i,j)^2$; e.g. the coefficient of $w_{10,1}^2$ is 29.2.
- Off diagonal terms are coefficients of cross terms $w(i,j)$; all zeros in this example.
- R-square indicates the goodness of fit.

Quadratic function without cross terms fits well for the majority of the metrics with R square higher than 0.96, but does not fit well for the lumpiness statistics.
- Add cross terms for lumpiness statistics.
Denote $f[k]$ as the quadratic function approximation for the k-th metric. User seek to optimize a combination of $f[k]$ subject to a set of constraints. Optimization can be either minimization or maximization.

The 9 issuance proportion variables are the choice variables.

User sets the weights that linearly combine metrics in the objective function and constraints.

Solving this optimization problem yields the optimal issuance strategy.
“Optimal” Issuance Strategy exists because all Treasury securities are not perfect substitutes.

- Model assumes that all securities are priced without market friction and no arbitrage opportunities are allowed.
- As such, ex ante, if we account for the present value of life-time cash flows in our cost calculation, we are indifferent.

Cost definition affects substitutability.

- Truncating cash-flows within a finite horizon introduces bias.
- PV or non-PV cash-flows makes a difference too.

The introduction of issuance constraints ensures that securities are not viewed as perfect substitutes.

- Must meet min/max constraint on each issuance.
- Must meet gradient constraint on change in consecutive new issuances.
- Should promote liquidity of all securities by penalizing non-stable issuance pattern.
- Each instrument follows its own issuance calendar.

Risk considerations also make the securities imperfect substitutes.

- Must control for rollover risk, for example, by meeting the requirement on portfolio WAM and/or lumpiness.
- Must control for interest risk, for example, by requiring reasonable stability in debt servicing cost.
- Must control for inflation risk, for example, by limiting the issuance proportion of TIPS.
- Must control for concentration risk, for example, by attributing lumpiness to various types of securities.

All these factors are policy choices.

- Cost within a finite horizon may be myopic. Terminal exposure helps to measure the burden for a potential buy-back.
- Issuance constraints reflect Treasury’s commitments: serving all clientele on the full curve and being regular and predictable.
- Risk considerations quantify Treasury’s degree of risk aversion.
Consider three ways of specifying “cost” to be minimized:

- (I) debt servicing cost during a 10-year horizon
- (II) debt servicing cost during a 20-year horizon, with the second 10-year period in a “run-off” mode
- (III) debt servicing cost during a 10-year horizon plus the outstanding amount at the end of year 10

<table>
<thead>
<tr>
<th>Year</th>
<th>Bill</th>
<th>Note</th>
<th>Bond</th>
<th>TIPS</th>
<th>10y cost $bn</th>
<th>20y cost $bn</th>
<th>10y outst. $bn</th>
<th>wam terminal yr</th>
<th>wam average yr</th>
<th>lumpiness $bn</th>
<th>cost vol $bn</th>
</tr>
</thead>
<tbody>
<tr>
<td>2012</td>
<td>42.83</td>
<td>48.78</td>
<td>4.46</td>
<td>3.92</td>
<td>6,506</td>
<td>11,284</td>
<td>15,078</td>
<td>7.40</td>
<td>6.18</td>
<td>5.73</td>
<td>281</td>
</tr>
<tr>
<td>2015</td>
<td>38.22</td>
<td>54.78</td>
<td>5.00</td>
<td>2.00</td>
<td>6,527</td>
<td>10,729</td>
<td>13,328</td>
<td>7.70</td>
<td>5.67</td>
<td>6.62</td>
<td>285</td>
</tr>
<tr>
<td>2018</td>
<td>50.22</td>
<td>45.78</td>
<td>2.00</td>
<td>2.00</td>
<td>5,000</td>
<td>5,000</td>
<td>10,000</td>
<td>5.00</td>
<td>5.00</td>
<td>5.00</td>
<td>100</td>
</tr>
<tr>
<td>2023</td>
<td>37.47</td>
<td>58.53</td>
<td>2.00</td>
<td>2.00</td>
<td>6,514</td>
<td>10,731</td>
<td>13,322</td>
<td>7.80</td>
<td>5.65</td>
<td>6.40</td>
<td>283</td>
</tr>
</tbody>
</table>

Considering the cost during the projection horizon only (Case I) leads to heavy use of coupons toward the end of the horizon, as only part of the coupon payments are counted as cost.

To mitigate the endpoint effect, either using a longer history (Case II) or accounting for the terminal exposure (Case III, in an approximate manner) seems to “correct” such myopic bias.

- In the first 5-year period, we act as if we are minimizing 10-year cost only.
- In the second 5-year period, we increase the use of Bills and drop the use of coupons.
Trade-off between Cost and Risk
Effect of Issuance and Risk Constraints

- Case (III) without any issuance constraint leads to a corner solution: 100% Bills issuance.
- Consider progressively adding risk constraints, with the objective of minimizing “lifetime” cost (as in Case III).
  - (IV) Both terminal WAM and path-wise average WAM need to greater than a certain level;
  - (V) In addition to (IV), lumpiness needs to below a certain level;
  - (VI) In addition to (IV) and (V), interest cost volatility should be below a certain level.
- We end up with the same optimal issuance strategy.
  - Because of the positive correlation between risk constraints and cost, limiting cost volatility and lumpiness in the constraints works in the same direction as minimizing cost.

<table>
<thead>
<tr>
<th>Year</th>
<th>Bill</th>
<th>Note</th>
<th>Bond</th>
<th>TIPS</th>
<th>10y cost $bn</th>
<th>20y cost $bn</th>
<th>10y oust. $bn</th>
<th>wam terminal yr</th>
<th>wam average yr</th>
<th>lumpiness $bn</th>
<th>cost vol $bn</th>
</tr>
</thead>
<tbody>
<tr>
<td>2012</td>
<td>42.83</td>
<td>48.78</td>
<td>4.46</td>
<td>3.92</td>
<td>6,785</td>
<td>12,044</td>
<td>15,197</td>
<td>10.74</td>
<td>8.00</td>
<td>8.23</td>
<td>289</td>
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<tr>
<td>2015</td>
<td>43.19</td>
<td>42.78</td>
<td>8.46</td>
<td>5.57</td>
<td></td>
<td></td>
<td></td>
<td>10.74</td>
<td>8.00</td>
<td>8.23</td>
<td></td>
</tr>
<tr>
<td>2018</td>
<td>40.00</td>
<td>47.43</td>
<td>10.00</td>
<td>2.57</td>
<td></td>
<td></td>
<td></td>
<td>10.74</td>
<td>8.00</td>
<td>8.23</td>
<td></td>
</tr>
<tr>
<td>2023</td>
<td>60.00</td>
<td>32.43</td>
<td>5.57</td>
<td>2.00</td>
<td></td>
<td></td>
<td></td>
<td>10.74</td>
<td>8.00</td>
<td>8.23</td>
<td></td>
</tr>
</tbody>
</table>

- Compared to Case (III), Bills are kept to lower cost.
- Notes are decreased in favor of Bond and TIPS (in that order). Although they are more expensive, they are needed to keep portfolio WAM above a certain level.
  - In 10 years, the interest cost is about $270 billion higher.
Appendix A: VAR Model Overview

The long term macro VAR model consists of two steps: calibration and simulation

Calibration

- Raw Inputs
  - Nominal yields of Treasuries
  - TIPS yields of Treasuries
  - CPI-Urban seasonally adjusted levels
  - Real GDP
  - Receipts and Outlays

- Raw Inputs -> Transformed Variables
  - Nominal yields -> Level, Slope and Curvature factors in Nelson-Siegel model
  - Break Even Inflation (BEI = nominal – real) at different maturity -> top two principal components, Spread1 and Spread2 (using principal component analysis)
  - Real GDP level -> monthly percentage change for stationarity
  - Receipts and Outlays -> normalized for stationarity and de-seasonalized

- Stack the variables (Level, Slope, Curv, %ΔGDP, %ΔCPI, Normalized-Rec, Normalized-Out, Spread1, Spread2) into a vector $X$
- $X_{t+1} = AX_t + b + \varepsilon_{t+1}$

where
- $A$ is a 9x9 regression matrix
- $b$ is a 9x1 vector of constants
- $\varepsilon_t$ are independent normal vectors, $N(0, \Sigma)$ distributed
- $\Sigma$ is a 9x9 covariance matrix
- We calibrate the model to historical data to estimate $A$, $b$, $\Sigma$
Appendix A: VAR Model Overview

- Simulation: we use the following equation:

\[
(X_{t+1} - \text{Proj}_{t+1}) = A(X_t - \text{Proj}_t) + \varepsilon_{t+1}
\]

where \( A, \Sigma \) are estimated in the calibration.

- The projected path \( \text{Proj}_t \) are transformed variables from OMB estimate, but user can specify the projected path.
- Convert the transformed simulated path \( X_t \) back into the underlying economic variables: interest rates, CPI and receipts and outlays.