TARP Warrants Valuation Methods
Robert A. Jarrow
September 22, 2009

Background and Summary

I was engaged as a contractor by the U. S. Treasury from July 15, 2009 to August 15, 2009 to assess the U.S. Treasury's TARP warrants valuation methodology. This document details my understanding of the Treasury's approach for valuing TARP warrants, gained from direct dialogue with Treasury staff members.

Under the Capital Purchase Program ("CPP"), the U.S. Department of the Treasury ("Treasury") received warrants in connection with each of its preferred stock investments in a Qualified Financial Institution ("QFI"). For investments in publicly traded institutions, Treasury received warrants to purchase common shares.[1] When a publicly-traded QFI repays Treasury's CPP preferred stock investment, the QFI is contractually entitled to repurchase the CPP warrants at fair market value.

The Treasury uses a number of different valuation approaches to help estimate fair market value. These approaches include indicative valuations from market participants, independent valuations from external asset managers, and modeled valuations using methodologies further described in this paper. The range of values provided in these approaches is analyzed by the Treasury to determine the adequacy of a QFI's assessments of fair market value.

Overview Warrant Repurchase Process under the CPP Contract

The warrant repurchase process is a multi-step procedure, starting with a QFI who wishes to repurchase the warrants submitting a determination of fair market value to Treasury. The Treasury can accept the fair market value or not. If the Treasury and the QFI cannot reach an agreement, either party may invoke an appraisal procedure. In this appraisal procedure, the bank and Treasury select independent appraisers. If these appraisers fail to agree, a third appraiser is hired, and subject to some limitations, a composite valuation of the three appraisals is used to establish the fair market value. If Treasury and the QFI cannot reach agreement regarding fair market value and neither party invokes the appraisal procedure, the Treasury intends to sell the warrants through an auction.

The Treasury has developed a robust set of procedures for evaluating initial QFI determinations based on three inputs: market prices (where available) and quotes from various market participants, financial models, and outside consultants/financial agents. The details of this repurchase process can be found at http://www.financialstability.gov/docs/CPP/Warrant-Statement.pdf**.**

Financial Modeling

The U.S. Treasury performs an in-depth model valuation as input to its assessment of a warrant's fair market value. The remainder of this report provides an in-depth description of the Treasury's valuation model.

To value its warrants, the Treasury uses a modified Black-Scholes model. For computations, the Treasury employs a binomial approximation to the Black-Scholes model. It is well known that the binomial model

[1] In the case of institutions that are not publicly-traded, Treasury received warrants to purchase preferred stock or debt and these warrants were exercised immediately upon closing the initial investment. As such, these warrants are no longer outstanding.

converges to the Black-Scholes model as the number of "steps" in the binomial's tree approaches infinity. The Black-Scholes model and its binomial approximation are well-accepted methods for pricing options by both academics and market participants (see Cox and Rubinstein [1985], Hull [2007], Jarrow and Turnbull [2000]).

An unadjusted (or not modified) Black-Scholes model for pricing equity options is based on the following simplifying assumptions: (1) no dividends, (2) constant interest rates, (3) the underlying stock's volatility is constant across time, and (4) frictionless markets (liquid markets and no funding costs). The U.S. Treasury uses a modified Black-Scholes model to incorporate the relaxation of these simplifying assumptions. The modifications employed are discussed below.

In addition, the unadjusted Black-Scholes model is formulated to price equity options and not warrants. Warrants differ from equity options in that when warrants are exercised, to fulfill the conditions of the warrant contract, the bank issues new shares. This is not the case with equity options. The U.S. Treasury's valuation method explicitly recognizes this distinction. This potential dilution effect of warrants is also discussed below.

The Standard Inputs

The standard inputs to the modified Black-Scholes warrant valuation model include the maturity date of the warrant, the warrant's strike price and the underlying stock price. The warrant's maturity date and strike price are as given in the CPP contract. For the current stock price, the Treasury uses a 20-day moving average of past stock prices to smooth any aberrations in the stock's price movements. However, the current stock price is also considered to include any recent shifts that may impact valuation.

The Modifications

1. Dividends

Unlike common stock, warrants are not entitled to dividend payments, and thus dividends reduce the value of the warrant by eroding the value of the underlying shares. The modified Black-Scholes model includes this dividend erosion by assuming that the underlying stock pays a constant dividend yield.

To estimate the dividend yield the Treasury analyzes the company's dividend payment history and investigates the company's implied or explicit dividend policies. The Treasury also examines recent dividend actions or market activity that may have changed dividend yields significantly. The effect of these changes is estimated and incorporated into the average dividend yield.

It is well known that with dividends, an American call option's value may differ from an otherwise identical European call option's value. This value difference is due to the possibility of early exercise. The TARP warrants can be exercised early; hence, they are American-type warrants. Early exercise is explicitly included within the binomial approximation procedure when valuing TARP warrants.

Justification

For a common stock, over a ten-year horizon, dividends will be stochastic and discrete. The Treasury approximates these discrete and stochastic dividend payments using a constant dividend yield. Since the underlying stock price is stochastic, a constant dividend yield implies that the total dividends paid over any quarter are stochastic. Hence, a constant dividend yield approximation incorporates the stochastic nature of these discrete dividend payments. This is a well-accepted approach to handling discrete and stochastic dividends (see Jarrow and Turnbull [2000, p. 258]).

2. Stochastic interest rates

The Treasury uses as the interest rate input the yield on a Treasury bond that matches the maturity of the warrant. Because the warrants in the Treasury portfolio are 9 to 10-year dated, the Treasury finds the appropriate matched maturity yield by straight-line extrapolating between the 7-year and 10-year constant maturity Treasury bonds.

Justification

It is well known (see Amin and Jarrow [1992]) that to modify the Black-Scholes formula for stochastic interest rates, there are two necessary adjustments. First, the yield on a Treasury bond matching the warrant's maturity should be used as the input to the Black Scholes formula. The Treasury incorporates this first adjustment. Second, when using historical volatility estimates, the volatility input should be adjusted to reflect the increased randomness due to the interest rate volatility and its correlation to the stock's return. This adjustment to the historic volatility is typically small and can often be excluded. However, when using implied volatilities, this second adjustment is unnecessary (see Jarrow and Wiggins [1989]). Because the Treasury uses an implied volatility estimation procedure whenever implied volatilities are available, the second adjustment is not used in the Treasury's valuation method.

3. Stochastic Volatility

There are two methods for estimating volatility: implied and historical. Without modification, both methods have limitations when estimating long-dated warrants with stochastic volatility. The Treasury uses a modified procedure involving both methods (where available) to construct a 10-year forward volatility curve. A forward volatility curve captures the stochastic nature of volatility. An "average" of the forward volatilities across this 10-year curve comprises the input to the modified Black-Scholes formula. Importantly, the Treasury also considers warrant values for a range of volatilities around this "average" forward volatility input.

For large financial institutions with liquid public equity and long-term options, the detailed procedure is as follows. The Treasury uses both observable implied volatility and historical volatility to construct a 10-year forward volatility curve. The initial segment of the curve consists of the observed implied volatilities for traded options. The last segment of the curve consists of a "normalized" 10-year average historical volatility. The volatility is normalized by removing any abnormally high recent volatilities from the estimate. The middle segment is determined using straight-line interpolation between the initial and terminal segments. The estimated forward volatility curve is typically downward-sloping, consistent with a reversion in volatilities to a long-run value.

Justification

It is well known (see Eisenberg and Jarrow [1994], Fouque, Papanicolaou and Sircar [2000]) that when volatilities are stochastic, a call option's value can be written as a weighted average of (constant volatility) Black-Scholes values (each with a different volatility input). The weights in this average correspond to the martingale probabilities of the different volatility inputs being realized. The volatility inputs are the average of the 10-year realized volatilities, i.e.

$$volatility\_input = \sqrt{\frac{\int_0^T \sigma_s ds}{T}}$$

where time 0 is today, time T is the maturity of the option (10 years), and σ_s is a possible realization of the random volatility at a future time s.

As discussed previously, the Treasury provides a range of Black-Scholes for various volatility inputs around the average of the 10-year forward volatility curve. These inputs can be interpreted as various possible averages of the future realized volatilities. The midpoint of this range is, therefore, an estimate for the option's value under a stochastic volatility model.[2]

3. Market Imperfections

The Treasury considers a number of market imperfections that could potentially cause the fair market value of a warrant to deviate from the model value (such as illiquidity of the warrant instrument or the bank's underlying equity). Judgment is used on a case-by-case basis to determine which adjustments, if any, for these market imperfections are appropriate.

Justification

Directly capturing market imperfections - market illiquidity and funding costs - in an option model is difficult (see Jarrow and Protter [2008], Broadie, Cvitanic and Soner [1998], Naik and Uppal [1994], and Cuoco and Liu [2000]). Each of these market imperfections can be considered as a type of transaction cost. It is well known that transaction costs make a market incomplete. In an incomplete market, there is a range of arbitrage free prices determined by the buying price (highest part of range) and a selling price (lowest part of range). The standard Black-Scholes value (without market imperfections) can be shown to lie between these two prices.

The buying and selling prices are determined by a trader's cost of replicating the identical cash flows to the warrant synthetically (for a long position and for a short position), including all market imperfections, via delta hedging. Note that with these market imperfections, there will be a buying premium and a selling discount reflecting the additional costs of obtaining the required cash flows.

Specific considerations with respect to market illiquidity and funding costs follow.

a. Illiquidity

The level of the stock price input into the Black-Scholes value captures the general impact of a depressed and/or illiquid market - making the warrants less valuable. This is distinct, however, from market liquidity considered as an endogenous transaction cost, i.e. a quantity impact on the price. A quantity impact on the price captures the notion that if you buy many warrants in an illiquid market, you need to pay more per share than if you buy only a single unit. Similarly, if you sell many warrants in an illiquid market, you will receive less per share than if you sell only a single unit.

The quantity impact on the market price is market-wide, and not trader-specific. For quoted prices (options prices for one round lot) or transaction prices (actual trading volume) the liquidity impact of the trade is captured by using an implied volatility (see Jarrow and Wiggins [1989]).

For large market trades of warrants, this component has to be separately included (as a discount to the model price) after the model's value is determined based on the Black Scholes formula using implied

---

[2] Of course, the midpoint assumes that each estimate of the realized future volatility provided is equally likely under the martingale probabilities.

volatilities. The magnitude of the potential discount is difficult to estimate. It depends on the size of the warrant position and the liquidity of the equity underlying the warrants. To access the magnitude of the liquidity impact, one can compute the number of shares an investor would short to "delta-hedge" the warrant position and compare that number to the average daily trading volume of the stock. The number of days of daily trading volume needed to delta hedge the position, across different banks, provides information on the relative liquidity of the warrant market.

It is important to emphasize that the price to the buyer (the bank) will exceed the price to the seller (the Treasury). The bank would pay a liquidity premium in buying (i.e. analogous to the "ask" in a bid-ask spread). The Treasury would incur a liquidity discount in selling (i.e. analogous to the "bid" in a bid-ask spread). Since the buyer and seller are meeting in a negotiated transaction, the "fair market" price should be in between the two prices.[3]

b. Funding Costs

Each bank has its own unique funding costs. These costs are determined by its existing balance sheet and credit worthiness. These unique funding costs determine the bank's internal cost of constructing a warrants cash flows synthetically (trading in the stock and borrowing/lending) via delta hedging.

In contrast, the fair market value is determined by the "marginal trader's" funding costs. The marginal trader is often not constrained, e.g. they can sell stock from inventory rather than shorting and incurring short sale fees. The marginal trader is the lowest cost transactor and their trades determine the fair market price. A market price that differs from the marginal trader's cost of construction - their buying/selling costs - will generate arbitrage opportunities for them. The marginal trader taking advantage of any such arbitrage opportunities will force the market price to reflect their funding costs.

There is significant empirical evidence that supports the claim that option models fit market prices well without explicitly including funding costs (see for example Pan [2002]). This supports the assertion that the fair market price is determined by marginal traders with small funding costs.

From the bank's perspective, if their funding costs are too high, for a long position, the bank would prefer to buy the warrants on the market rather than creating the same cash flows synthetically on their balance sheet via delta hedging. Similarly, for a short position, the bank would prefer selling the warrants in the market rather than holding the warrants on their balance sheet and shorting the warrants synthetically via delta hedging.

The Treasury's mandate is to obtain the "fair market" value of the warrants. Consequently, the marginal trader's funding costs are those that are relevant, not the bank's. Since the modified Black-Scholes model captures dividends, stochastic interest rates, and stochastic volatility, the only remaining considerations are market liquidity[4] and funding costs. If one uses the Black-Scholes model (as modified above) without market liquidity and funding costs, and (for a given future realized volatility) if one computes an option's implied volatility, then the implied volatility will incorporate both the historic volatility and the marginal trader's liquidity impact and funding costs (see Jarrow and Wiggins [1989]).

---

[3] Note that the Black-Scholes value using implied volatilities is near the midpoint of the range determined by the buying premium and selling discount.

[4] The adjustment for market liquidity – a quantity adjustment on the price - is given by a scale adjustment after the model's value is computed using the implied volatilities which are based on typically small transaction volumes for the traded options.

For this reason, using the implicit volatility is key to including market liquidity and funding costs into the valuation procedure.

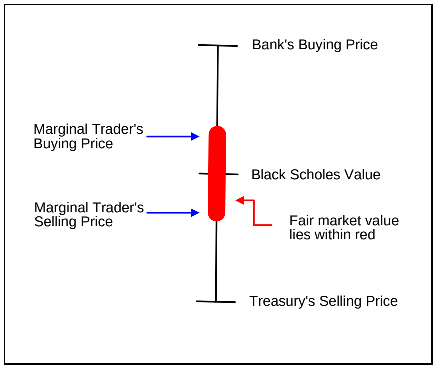

Figure 1: Various Warrant Prices

To understand the difference between the various prices, consider the following alternatives as reflected in Figure 1.

i. Bank's buying price: The bank can keep the warrants on its balance sheet, but borrow and trade in the underlying stock to remove the economic impact of the warrants. Note that the bank is short the warrants to the Treasury. Hence, it has to synthetically create a long position in the warrants via delta hedging, i.e. it has to buy the stock and borrow. When borrowing, the bank is incurring its higher funding costs (through a higher borrowing rate). It can be shown that the cost of synthetic construction - the bank's buying price - exceeds the Black Scholes value without the inclusion of these funding costs.[5]

ii. Marginal trader's buying price: The fair market price is determined by the marginal trader's funding cost for a long position in the warrant. The marginal trader's funding costs are less than those of the bank's (see Figure 1).

iii. Treasury's selling price: The Treasury has the analogous decision to the bank's. It can keep the warrants on its balance sheet and remove their economic risk by delta hedging. If the Treasury keeps the warrants on its balance sheet, it needs to sell the underlying stock and lend cash. The Treasury's funding costs are the relevant consideration here. The lending rate is the Treasury rate (zero funding cost). Although there are no funding costs, the Treasury's selling price would include short sales fees and the liquidity discount. Short sales fees and the liquidity discount make the selling price below the Black Scholes value without the inclusion of these costs (see Figure 1). It is important to note that the bank's selling price would be approximately the same as the

[5] Of course, the inclusion of liquidity costs in the bank's delta hedging will further increase the buying price due to the liquidity impact on the price.

Treasury's. The reason is that the bank's lending rate is the Treasury rate as well (zero funding cost). [6]

iv. Marginal trader's selling price: The fair market price is determined by the marginal trader's funding cost for a short position in the warrant. The marginal trader's funding costs are less than the bank's (see Figure 1).

v. Fair market price: The fair market price lies between the marginal trader's buying and selling prices. It is determined by market equilibrium considerations.

A negotiated transaction between the bank and Treasury has as the range for negotiation all prices between the bank's buying price and the Treasury's selling price. The Black-Scholes value without the inclusion of funding costs lies between these two. The fair market value is also between the two and determined by the marginal trader's funding costs. The Treasury's mandate is to obtain the "fair market" value of the warrants. Consequently, if Treasury's model were the only valuation mechanism is use, starting discussions regarding value with the modified Black-Scholes value as computed above would be appropriate.

Dilution from Warrant Exercise

Warrants differ from standard equity options in that the shares that a warrant holder receives, if exercised, are issued by the company and are not currently outstanding. As such, the exercise of the warrants triggers an issuance of shares by the company, and a potential dilution of existing shareholders' values. The Treasury makes no adjustment to the Black-Scholes value for this dilution effect, due to the fact that this dilution effect is rationally anticipated by market participants and already included in the current stock price input into the modified Black-Scholes formula.

Justification

There are two issues that arise due to warrant exercise resulting in the issuance of new shares. These are called sequential exercise and strategic exercise. Sequential exercise occurs when a large trader or monopolist owns most of the shares and they can create more value by exercising sequentially rather than as a block (see Constantinides [1984], Emanuel [1983], Spatt and Sterbenz [1988], and Linder and Trautmann [2009]). Strategic exercise occurs when a large trader or a group of small traders (acting independently via a Nash equilibrium) can create more value by exercising the shares only if the market price exceeds a value greater than the strike price (see Cox and Rubinstein [1985], p. 396, Galai and Schneller [1978], and Crouchy and Galai1[1994]). This is due to a potential transfer of wealth from shareholders to the liability holders due to dilution and an inflow of cash into the firm.

Since the Treasury is mandated to determine a fair market price, sequential exercise is not a relevant consideration because it only applies to a monopolist. Strategic exercise is a potential consideration, but to obtain a realistic representation of both the dilution and cash inflow effects, one must explicitly model

---

[6] There is no economic rationale for why the bank's high funding costs (higher borrowing rate) should be used as a justification for obtaining a lower selling price. The logic underlying using the bank's selling price is that the bank wants to buy back the warrants from the Treasury while simultaneously creating a short position in the warrants to finance the purchase (using the proceeds from the short position). This argument effectively retains the economic position of the outstanding warrants on the bank's balance sheet.

the liability structure of the bank. Given the complexity of a large financial institutions balance sheet (including off and on balance sheet items), this is an impossible task.[7]

An alternative approach, consistent with both theory and empirical evidence (see Schulz and Trautmann, [1994] is to assume that the market rationally anticipates the dilution and cash flow effects, and that these are embedded in the current stock price input into the modified Black-Scholes formula. Note also that the adjustment for stochastic volatility is consistent with this implementation. This is the approach that the Treasury adopts.

Warrant Contract Terms

The terms of the Treasury's warrants are specified in the Form of the Warrant documentation available on www.financialstability.gov. Certain of these terms can affect the warrant's value in a way not captured by an unadjusted Black-Scholes model. The Treasury considers each of these effects and includes them, when possible, in the valuation of the warrants.

Dividend protection. The warrant document (see http://www.financialstability.gov/docs/CPP/warrant.pdf) specifies protective adjustments to the terms of the warrants in the case that dividends in excess of certain levels are paid. This dividend protection would never decrease and would sometimes increase the value of the warrant. The exact effect on the value of the warrant depends on many factors including dividends at the time of funding, current dividends, and expected future dividend activity.

Business combinations. The warrant document also specifies certain adjustments to be made under certain business combinations. The effect of the terms is that some out-of-the-money warrants could become worthless (i.e. lose all their time value) if the underlying equity is purchased for cash by another company. Business combinations could also change the volatility of the underlying equity of a warrant.

Conclusion

As documented above, it is my belief that the Treasury's modeling methodology for valuing the warrants is consistent with industry best practice and the highest academic standards. The methodology uses the industry standard model for pricing options, the Black-Scholes model, in a modified form to account for the size of the warrant position, stochastic interest rates, stochastic volatility, as well as numerous market imperfections.

Furthermore, as previously detailed, the Treasury's financial model is only one component of a robust valuation procedure. For warrant positions that are evaluated, the Treasury also collects market prices (where available) or indications from market participants and valuations from outside consultants/financial agents. All valuation information is considered in the determination of an appropriate fair market value for the warrants of a specific institution.

The valuation process results in a warrant valuation that is fair to both the participating banks and the U.S. taxpayers.

---

[7] Note that the existing academic literature only considers too simple and unrealistic capital structures.

References

K. Amin and R. Jarrow, 1992, Pricing options on risky assets in a stochastic interest rate economy, *Mathematical Finance*, 2, 4, 217-237.

M. Broadie, J. Cvitanic and H. Soner, 1998, Optimal replication of contingent claims under portfolio constraints, *Review of Financial Studies*, 111, 59 – 79.

J. Cox and M. Rubinstein, 1985, *Options Markets*, Prentice Hall.

G. Constantinides, 1984, Warrant exercise and bond conversion in competitive markets, *Journal of Financial Economics*, 13, 371 – 397.

M. Crouchy and D. Galai, 1994, The interaction between the financial and investment decisions of the firm: the case of issuing warrants in a levered firm, *Journal of Banking and Finance*, 18, 861 – 880.

D. Cuoco and H. Liu, 2000, A martingale characterization of consumption choices and hedging costs with margin requirements, *Mathematical Finance*, 10, 3, 355- 385.

L. Eisenberg and R. Jarrow, 1994, Option pricing with random volatilities in complete markets, *Review of Quantitative Finance and Accounting*, 4, 5-17.

D. Emanuel, 1983, Warrant valuation and exercise strategy, *Journal of Financial Economics*, 12, 211 – 235.

J. Fouque, G. Papanicolaou and K.R. Sircar, 2000, *Derivatives in Financial Markets with Stochastic Volatility*, Cambridge University Press.

D. Galai and M. Schneller, 1978, Pricing of warrants and the value of the firm, *Journal of Finance*, 33, 1333 – 1342.

J. Hull, 2007, *Options, Futures and Other Derivatives*, 5th edition, Prentice Hall.

R. Jarrow and P. Protter, 2004, Structural versus reduced form models: a new information based perspective, *Journal of Investment Management*, 2 (2), 1 - 10.

R. Jarrow and P. Protter, 2008, Liquidity Risk and Option Pricing Theory, *Handbooks in OR&MS*, vol 15, eds. J Birge and V. Linetsky, Elsevier.

R. Jarrow and S. Turnbull, 2000, *Derivative Securities*, 2nd edition, Southwestern Publishers.

R. Jarrow and J. Wiggins, 1989, Option pricing and implicit volatilities, *Journal of Economic Surveys*, 3, 1, 59 – 81.

T Linder and S. Trautmann, 2009, Sequential Warrant Exercise in Large Trader Economies, working paper, Johannes Gutenberg-Universitat, Mainz, Germany.

V. Naik and R. Uppal, 1994, Leverage constraints and the optimal hedging of stock and bond options, *Journal of Financial and Quantitative Analysis*, 29 (2), 199 – 222.

J. Pan, 2002, The jump-risk premia implicit in options: evidence from an integrated time-series study, *Journal of Financial Economics*, 63, 3-50.

U. Schulz and S. Trautmann, 1994, Robustness of option-like warrant valuation, *Journal of Banking and Finance*, 18, 841 – 859.

C. Spatt and F. Sterbenz, 1988, Warrant exercise, dividends and reinvestment policy, *Journal of Finance*, 43, 493 – 506.